CS 58500 Theoretical Computer Science Toolkit (Spring 2026) Problem Set # 2
{{Your name}} **Due: Mar 2nd**

The questions have been labeled with the date of the lecture in which the relevant material is covered.

Problem 1 (02/12, 02/17). (30 pts)

1. For a matrix A, vector b, positive scalars $t_1, t_2 \geq 0$, and convex functions f_1 and f_2, the function $g(x) = t_1 f_1(Ax + b) + t_2 f_2(x)$ is convex.
2. Show that the intersection of convex sets is convex; show that for convex functions, f, g, the function $h(x) = \max\{f(x), g(x)\}$ is also convex.
3. Show that for any $t \geq 0$, the level set $L(t) := \{x : f(x) \geq t\}$ of a logconcave function f is convex.

Solution.

Problem 2 (02/19). (20 pts)

Prove the following Theorem of Bieberbach from 1915: Let $K \subseteq \mathbb{R}^n$ be a compact set. Then

$$\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(B_2^n) \left(\frac{\mathrm{diam}(K)}{2}\right)^n$$

where $\mathrm{diam}(K) := \max\{\|x - y\|_2 : x, y \in K\}$.

1. Show that $\mathrm{diam}(\mathrm{conv}(K)) = \mathrm{diam}(K)$. So, we can assume that K is convex.
2. For any unit vector $u \in \mathbb{S}^{n-1}$, let $S_u(K)$ be the Steiner symmetral of K along u. For each $x \in u^\perp := \{y \in \mathbb{R}^n : \langle y, u\rangle = 0\}$, the length of the segment $K \cap (x + \mathbb{R}u)$ equals the length of $S_u(K) \cap (x + \mathbb{R}u)$. Steiner symmetrization can preserve convexity and volume. In this sub-problem, you need to prove that Steiner symmetrization cannot increase the diameter:

   $$\mathrm{diam}(S_u(K)) \le \mathrm{diam}(K)$$

   Hint. Consider $M = \frac{K + \rho K}{2}$, where ρ is the reflection across the hyperplane $u^\perp$.
3. Show that sequentially applying Steiner symmetrizing w.r.t. the standard basis ($u_1 = e_2, u_2 = e_2, \ldots, u_n = e_n$) makes the set centrally symmetric (i.e., $K = -K$).
4. Prove the volume bound in the Bieberbach Theorem.

Solution.

Problem 3 (02/12, 02/24). (30 pts)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable and convex. We say f is L-smooth if

$$|\nabla f(x) - \nabla f(y)| \leq L\|x - y\|_2 \quad \forall x, y \in \mathbb{R}^n$$

1. Prove that f is L-smooth if and only if

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2}\|y - x\|_2^2 \quad \forall x, y \in \mathbb{R}^n$$

2. If f is twice-differentiable, f is L-smooth if and only if

$$|\nabla^2 f(x)[v, v]| = |v^\top \nabla^2 f(x) v| \leq L\|v\|_2^2 \quad \forall x, v \in \mathbb{R}^n$$

Solution.

Problem 4. (20 pts)

The spanning tree polytope of an undirected graph $G = (V, E)$ is given by

$$P = \{x \in \mathbb{R}^{|E|}_{\geq 0} : \sum_{e \in E} x_e = |V| - 1, \sum_{(u,v) \in E \cap (S \times S)} x_{u,v} \leq |S| - 1 \quad \forall S \subseteq V, |S| \geq 1\}$$

Then, the extreme points of this polytope exactly correspond to the indicator vectors of spanning trees of G.

In this problem, you need to design a membership oracle for this polytope. That is, for any input vector $x \in \mathbb{R}^n$, decide whether $x \in P$ in poly$(|V|, |E|)$ time.

Solution.